Strongly localized modes in discrete systems with quadratic nonlinearity

S. Darmanyan,[1,2] A. Kobyakov,[1] and F. Lederer[1]
[1]Institute of Solid State Theory and Theoretical Optics, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, 07743 Jena, Germany
[2]Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow Region 142092, Russia
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We report the existence of bright and dark families of strongly localized modes in discrete systems with a quadratic nonlinearity. It is shown analytically and confirmed numerically that the second-harmonic field may form stable bound states with fundamental fields of different topologies. Furthermore, we found different types of solutions having analogs neither in other discrete models nor in continuum models and studied the background stability of dark modes. [S1063-651X(98)03902-6]

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Since the original investigations [1–6] a considerable and steadily growing amount of interest has been centered on the study of intrinsic strongly localized modes (SLMs) in nonlinear discrete systems because of their relevance in various fields, e.g., solid-state physics, optics, and biology [7–13]. The dynamics of many systems is described by the discrete nonlinear Schrödinger equation (DNLSE) or by its modifications. Prominent examples are lattices with different nonlinear potentials as well as arrays of linearly coupled optical waveguides with a cubic nonlinearity. Bright and dark SLMs may exist in this environment [7–9,12,13] where bright ones are formed due to modulational instabilities (MI) of stationary nonlinear solutions. Dark SLMs, on the contrary, need a modulationally stable background. The experimental demonstration of these phenomena was reported in [14,15].

Within the past several years a renewed interest has emerged in nonlinear systems where the dynamics of two fields can be described by two Schrödinger equations coupled by a quadratic nonlinearity. With respect to the continuous case, representative examples are the envelope evolution of an optical field in bulk media or in film waveguides with a quadratic nonlinearity [16,17] as well as the dynamics of long-wavelength excitations at the interface between two organic crystals [18,19]. The particular form of nonlinearity leads to energy exchange between the two field components and additionally brings another crucial parameter, the phase mismatch, into play. Although these systems are not integrable, stable mutually locked solitary waves may exist in continuous media, which was experimentally confirmed in optics [17].

At the present time, quickly developing technologies such as epitaxial growth, ion exchange in solids, and electrical poling allow for fabrication of different kinds of thin films, multilayer structures, and arrays of optical waveguides for advanced photonics applications. The discrete nature of such structures is responsible for qualitatively different types of excitations and effects connected with them. In particular, the so-called Fermi-resonance interface modes were shown to appear owing to the Fermi resonance between excitations of molecules situated at the interface between two organic molecular crystals. Moreover, the above-mentioned excitations may form bound states of different symmetries and demonstrate interesting nonlinear properties such as bistability and thus represent promising laboratories for future generation of optoelectronic devices [18,20,21].

Another prominent example of discrete systems represents arrays of coupled optical waveguides. Applications of localized modes in such systems for all-optical information processing were discussed in [12,13]. We mention that only cubic (or $\chi^{(3)}$) nonlinearity of waveguides has been considered, although quadratically nonlinear media provide a much greater variety of effects [22] that, more importantly, are obtainable for lower power compared to the $\chi^{(3)}$ scenario.

A relevant question not addressed until now is how discreteness affects the nonlinear dynamics in systems with quadratic nonlinearity far beyond the continuum limit. Thus the aim of this paper is to demonstrate the existence of families of bright and dark highly localized two-field states and to study their fundamental properties.

The evolution of the two-component field in a discrete quadratic medium may be described by nonlinearly coupled difference-differential equations as

$$\begin{aligned} i\,\frac{dA_n}{dz}+c_a(A_{n+1}+A_{n-1})+2\gamma A_n^* B_n=0,\\ i\,\frac{dB_n}{dz}+c_b(B_{n+1}+B_{n-1})+\beta B_n+\gamma A_n^2=0. \end{aligned} \tag{1}$$

If an array of optical waveguides is concerned A_n and B_n represent the fundamental frequency (FF) and the second-harmonic (SH) amplitudes in the nth waveguide, respectively. Similarly, they designate the respective vibration amplitudes of the nth molecule at a molecular interface. The evolution variable z denotes either the propagation distance along the waveguides or the time in the vibration process, $c_{a,b}$ and γ are the linear and nonlinear coupling coefficients, respectively, and β is either the wave-vector mismatch or the detuning from the Fermi resonance. Equations (1) were conveniently normalized by using both a characteristic length (or time) and amplitude.

It is evident that in the long-wavelength case (slow variation with n) the dynamic equations (1) transform into the continuum limit. Several types of localized solutions to this system, viz., bright, dark (so-called twin-hole), and semidark solitary waves have been discussed in optics and solid-state physics [16–19,23]. This gives some evidence that similar localized solutions may exist in the discrete case as well. However, until now it had not been clear which shape the continuum solitary-wave solutions were going to attain for asymptotically diminishing width, i.e., for strongly localized modes.

Prior to the study of SLMs, it was useful to look for the stationary plane-wave solutions to Eqs. (1). Inserting the ansatz $A_n(t)=a\ \exp[i(qn-kz)]$, $B_n(t)=b\ \exp[2i(qn-kz)]$ into Eqs. (1), we get a relation between the FF and SH amplitudes a and b as

$$a^2=4b^2+\frac{b}{\gamma}\,(4c_a\cos q-2c_b\cos 2q-\beta)>0. \tag{2}$$

As a matter of fact, a certain SH amplitude b applies to a FF amplitude a of either sign. This behavior will be encountered for SLMs too. The dispersion law, which relates the wave vector in propagation direction (k) with the transverse wave vector (q) and SH amplitude b,

$$k=-2(\gamma b+c_a\cos q), \tag{3}$$

shows that the nonlinear shift $-2\gamma b$ determines the wave vector of the nonlinear mode provided the nonlinear coupling exceeds the linear one ($|\gamma b|\gg c_a$). As it will be shown below, this very condition is a prerequisite to the formation of SLMs.

To identify SLMs we extend an approach used by Page [6] for one-component vibrations in nonlinear lattices toward the two-field case. Because we are concerned only with resting solutions, i.e., $q=0,\pi$, we may write $A_n=a_n\exp(-ikz)$, $B_n=b_n\exp(-2ikz)$, where a_n,b_n can attain either sign. As usual, we may classify SLMs as symmetric or antisymmetric bright or dark states localized mainly at a single (odd mode) or two (even mode) sites.

To obtain the amplitudes at these sites and for the nearest neighbors we impose the respective symmetry on the ansatz. To this end we introduce amplitudes ρ_n and μ_n normalized by the maximum value a_p,b_p, that is, $a_n=a_p\rho_n$, $b_n=b_p\mu_n$, and indicate odd and even modes by $p=o,e$, respectively. As mentioned previously, strong nonlinear coupling is assumed, which allows the introduction of small parameters such as

$$\varepsilon_p=-c_a/k\approx c_a/2\gamma b_p,$$

$$\delta_p=-c_b/(2k+\beta)\approx c_b/(4\gamma b_p-\beta). \tag{4}$$

Seeking an odd symmetric bright SLM as a solution with the vanishing value of $\rho_{\pm n},\mu_{\pm n}$ for $n\geq 2$ and $\rho_{-1}=\rho_1$, $|\rho_1|\ll\rho_0=1$, $\mu_{-1}=\mu_1$, and $|\mu_1|\ll\mu_0=1$, we insert the ansatz for A_n and B_n into Eqs. (1) and solve the resulting system of algebraic equations. Neglecting second-order terms in the small parameters, we get

$$a_o^2\approx 4b_o^2-\beta b_o/\gamma>0,\quad \rho_1\approx\varepsilon_o,\quad \mu_1\approx\delta_o, \tag{5}$$

where the SH amplitude b_o is a free parameter. Note that an extremely high localization (at only one site) of the FF or SH field may occur if the linear coupling is very small (c_a or c_b tend to vanish). The method used permits the calculation of SLM amplitudes with any prescribed accuracy. Here, in focusing on the physical aspects of these phenomena we restrict ourselves to the first-order approximation.

Accordingly, an even bright SLM looks like $\rho_{\pm n}\approx 0$, $\mu_{\pm n}\approx 0$ for $n\geq 3$ and $|\rho_{\pm 2}|\ll\rho_{-1}=1$, $|\mu_{\pm 2}|\ll\mu_{\pm 1}=1$, $\rho_1=s$, where $s=1$ and -1 correspond to the symmetric and the antisymmetric modes, respectively. The amplitudes are given by

$$a_e^2\approx 4b_e^2+b_e(2sc_a-c_b-\beta)/\gamma>0,\quad \rho_{-2}\approx\varepsilon_e,$$

$$\rho_2=s\rho_{-2},\quad \mu_{\pm 2}\sim\delta_e. \tag{6}$$

For obvious symmetry reasons the subscript $n=0$ was abandoned for even modes. In looking at Eqs. (5) and (6) it is evident that the ratio $\beta/\gamma b>0$ has an upper bound for SLMs to exist.

Thus odd symmetric and even symmetric as well as antisymmetric SLMs can be identified from Eqs. (4)–(6). Note that for even solutions four different types of the FF field correspond to one SH pattern.

A characteristic feature of nonlinear discrete systems consists in that the excitation induces an effective periodic potential similar to the Peierls-Nabarro (PN) potential (see, e.g., [2,8,9] and references therein), which breaks the translational invariance and may thus prevent the dislocation of SLMs. Moreover, the PN potential was frequently used to predict the stability behavior of a particular SLM [9]. Thus the identification of the PN barrier between two different SLMs is a relevant subject to be addressed here.

Equations (1) exhibit two integrals of motion that represent the total intensity and the Hamiltonian and can be written as

$$I=\sum_n\ (|A_n|^2+2|B_n|^2), \tag{7}$$

$$H=-\sum_n\ (c_aA_nA_{n+1}^*+c_bB_nB_{n+1}^*+\gamma A_n^2B_n^*+\tfrac{1}{2}\beta|B_n|^2+\text{c.c.}). \tag{8}$$

Even and odd SLMs of equivalent topology may be considered as realizations of a common mode centered either at or in between the array (lattice) elements [8]. Thus we have to require the same intensity (7) for both SLMs, which results in a relation between b_o and b_e. Using this relation, the difference between the Hamiltonians of the odd (H_o) and the even (H_e) SLM can be calculated. This difference is likewise the PN barrier that separates both realizations from each other. As a matter of fact if the PN barrier is nonzero and the transverse wave vector q is less than a certain critical value the SLM is at rest. For cubic nonlinearities it has been shown that stability requires a minimum for in-phase modes or a maximum of the PN potential for out-of-phase, or staggered, modes [9]. This change of the stability criterion reflects the symmetry properties of the relevant dynamic equation and the Hamiltonian. If a pure on-site cubic nonlinearity is concerned, in-phase SLMs may only exist if the nonlinearity is positive where a negative nonlinearity supports out-of-phase

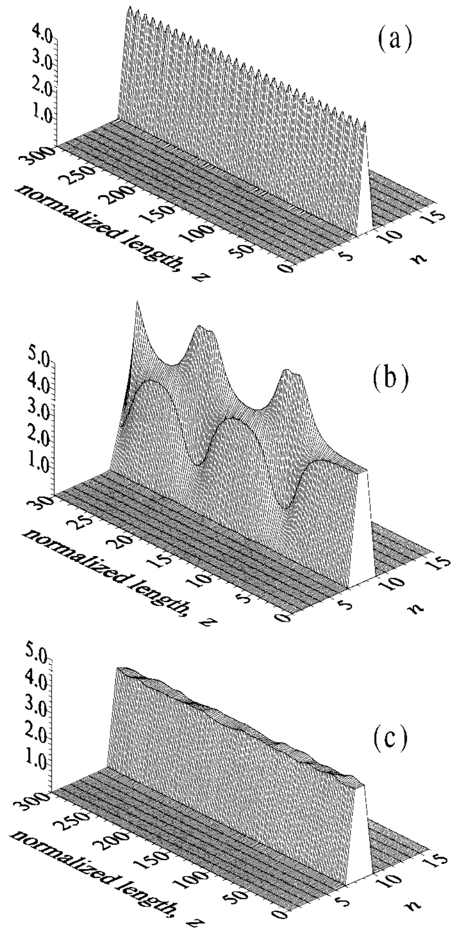

FIG. 1. Evolution of the normalized intensity of bright SLMs. The FF component is shown and the SH field has a similar form. (a) A stable odd SLM, (b) a slightly antisymmetrically perturbed even SLM consisting of both unstaggered FF and SH components, and (c) a strongly (symmetrically and antisymmetrically) perturbed even SLM consisting of twisted FF and unstaggered SH components. The parameters are $c_a=c_b=0.15$, $\beta=0$, $\gamma=1$, and $b=1$.

SLMs. Because the sign of the PN barrier also changes, odd SLMs are stable for any sign of nonlinearity [9].

For quadratic nonlinearities dynamics Eqs. (1) lack the above symmetry and a stable solution requires a minimum of the PN potential. In turn, it can be shown that $H_o<H_e$ holds always regardless of the sign of the nonlinearity (γb). Hence odd solutions are expected to be stable, unlike unstaggered and staggered even modes, which are unstable. This was indeed confirmed by direct numerical calculations presented in Figs. 1(a) and 1(b) (for convenience excitations are enumerated by positive numbers). A stable propagation of the perturbed odd SLM as well as a decay of the even mode (both FF and SH fields are unstaggered), which immediately transforms to an oscillatory state, can be clearly seen.

We mention that for $\gamma b>0$ even antisymmetric and for $\gamma b<0$ even symmetric SLMs are neither unstaggered nor staggered solutions and can be obtained from those by changing the phase of excitations on sites $n\geqslant 1$ by π. Hence we called these modes *twisted* (unstaggered and staggered). Such SLMs in the systems described by the DNLSE have been shown to exhibit quite interesting properties [24]. In particular, unlike the ''traditional'' even unstaggered and staggered modes they are stable, but only above some critical localization degree [24], which explains the absence of twisted solitonlike solutions in the continuum limit. It is worth realizing that for large mismatch Eqs. (1) transform to the DNLSE and therefore twisted SLMs could also be counted among the solutions of Eqs. (1). However, the two-field modes in discrete media with quadratic nonlinearity represent a pair of *twisted* FF and *unstaggered* SH fields. As can be inferred from Fig. 1(c), such a topology drastically influences stability properties of the SLM. The twisted fundamental field tends to stabilize the whole mode, which, being much more strongerly perturbed compared than the unstaggered SLM [Fig. 1(b)], demonstrates rather stable behavior.

Now a remark is in order. Because we have considered the case of small coupling constants, it is worth mentioning two additional approaches applied to study dynamics of localized solutions in other discrete and continuous systems. These are (i) the so-called semiclassical or dispersionless limit of partial differential equations ([25] contains a number of papers on this subject) and (ii) the ''anticontinuous'' limit, which corresponds to asymptotically vanishing coupling in discrete systems (see, e.g., [26] and references therein). The first method is exclusively applicable to continuous systems and thus not so relevant to the *strongly localized* modes we deal with. We only note that localized modes initially excited in the continuous system with the infinitesimally small dispersion would spread, demonstrating rather complicating dynamics [25]. As far as the second approach is concerned, the concept of an anticontinuous limit has been used to prove the existence of SLMs for one-field time-reversible or Hamiltonian discrete systems [26].

Now we proceed in looking for dark SLMs. To this end we substitute $A_n=a\varphi_n\exp(-ikz)$ and $B_n=b\psi_n\exp(-2ikz)$ into Eqs. (1), where a and b are taken from the stationary plane-wave solution (2) and use the same small parameters (4) but drop the subscript p. The odd symmetric dark mode can be written as $\varphi_{\pm n}=\psi_{\pm n}=1$ for $n\geqslant 2$ and

$$\varphi_0\approx 2\varepsilon,\quad \varphi_{\pm 1}\approx 1+(\varepsilon+\delta)/2,\quad \psi_0\approx 2\delta,\quad \psi_{\pm 1}\approx 1+\varepsilon. \tag{9}$$

We note that the shape of the SH part of this dark solution does not change for the antisymmetric or kinklike mode for the FF wave [$\varphi_0=0$, $\varphi_{-n}=-\varphi_{+n}$, where φ_{+n} is given by Eq. (9)].

The even dark mode looks like $\varphi_{\pm n}=\psi_{\pm n}=1$ for $n\geqslant 3$ and

$$\varphi_{\pm 1}\approx\varepsilon,\quad \varphi_{\pm 2}\approx 1+(\varepsilon+\delta)/2,\quad \psi_{\pm 1}\approx\delta,\quad \psi_{\pm 2}\approx 1+\varepsilon. \tag{10}$$

The corresponding kinklike mode for the FF field is also described by Eq. (10), where $\varphi_{-n}=-\varphi_{+n}$. Note that the amplitudes of dark out-of-phase $(q=\pi)$ SLMs are likewise given by Eqs. (9) and (10) provided one changes the sign of c_a together with the transformation $a_n\rightarrow(-1)^n a_n$. It turned out that SLM shapes obtained by direct numerical calculations are in good agreement with Eqs. (4)–(6), (9), and (10).

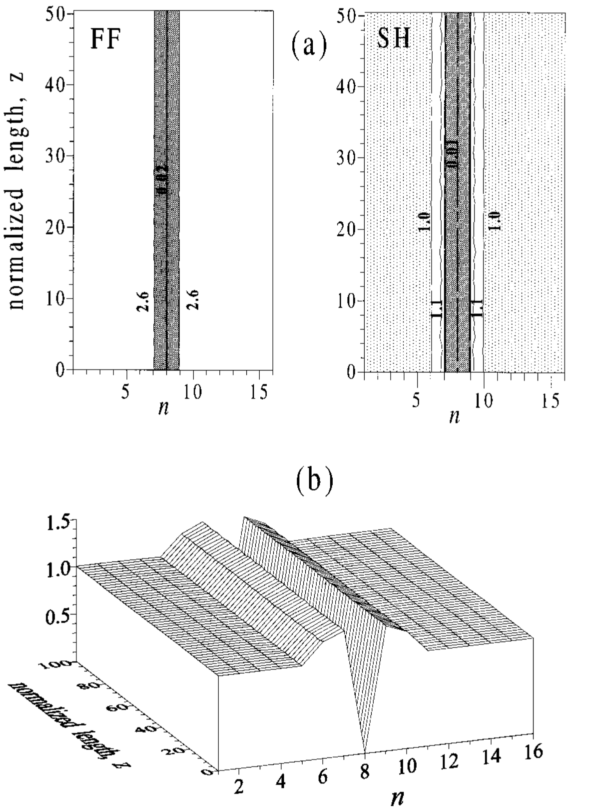

FIG. 2. Evolution of the normalized intensity of out-of-phase odd dark SLMs: (a) a SLM where a single-site localized FF is locked to a double-hump SH component and (b) a double-hump dark SLM where excitation is spread over five sites (the corresponding FF field has the same shape). The parameters are $c_a = c_b = 0.1$, $\beta = -2$, and $b = -1$.

We mention that a few SLMs derived have their respective counterparts in continuum media, whereas most of them are unique for discrete systems. For example, the extremely localized dark mode, where the FF amplitude differs only at a single site (odd mode) or at two sites (even mode) from the uniform background, may form provided $b = \beta c_a [2\gamma(2c_a + c_b)]^{-1}$; see Eqs. (4), (9), and (10). Such an odd mode is plotted in Fig. 2(a). Note the peculiar shape of the SH field when the amplitude $\psi_{\pm 1}$ exceeds the background amplitude $\psi_{\pm n}$, $n \geqslant 2$. This corresponds to the formation of the double-hump dark solutions, which may also exist for both the FF and the SH fields and for the localization at more than three sites. Propagation of such a mode where five channels are involved is shown in Fig. 2(b) for the SH field. As can be seen, numerical calculations demonstrate an appreciable robustness of these modes upon propagation.

Apart from SLMs discussed above, we have also observed gray, antidark (or bright on a nonzero background) modes both for the SH and FF waves as well as some exotic ones, e.g., a symbiotic pair of a gray FF wave and a double-hump antidark SH wave. As an example, stable propagation of the antidark SLM is presented in Fig. 3.

The stability of dark SLMs is essentially determined by the stability of the background against spatial modulations. So it has been shown that in the continuum case the dark solitary wave is unstable due to MI of the background [23].

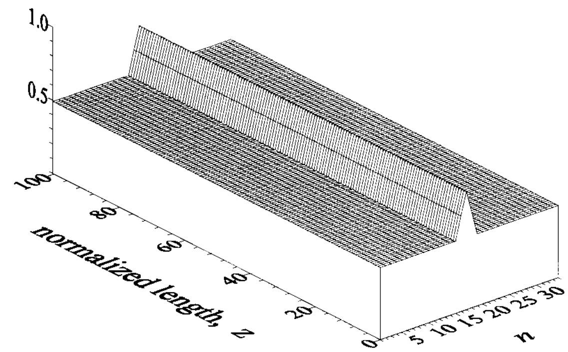

FIG. 3. Stable propagation of an antidark SLM (the normalized SH intensity is shown). The corresponding FF field has the same shape. The parameters are $c_a = c_b = 0.1$, $\beta = -1.8$, $b = -0.7$, and $q = \pi$.

Thus it is in order to check the dark SLMs derived above against MI.

To do this the stationary plane-wave solution [see Eqs. (2) and (3)] is slightly modulated: $a \to a + \xi_n(t)$ and $b \to b + \zeta_n(t)$, where $\xi_n = u_1 \exp(i[Qn - Kz]) + u_2 \exp(-i[Qn - K^* z])$ and $\zeta_n = v_1 \exp(i[Qn - Kz]) + v_2 \exp(-i[Qn - K^* z])$. By inserting this into Eqs. (1) and performing a linear stability analysis, we end up with a characteristic equation for K. For the situation we are concerned with here $(q = 0, \pi)$ this equation reduces to $K^4 + \alpha_2 K^2 + \alpha_0 = 0$, with

$$\alpha_2 = 4\gamma^2 (b^2 - 2a^2) - f^2 - g^2,$$
$$\alpha_0 = (fg)^2 - (2\gamma b g)^2 - 8(\gamma a)^2 fg + (2\gamma a)^4, \tag{11}$$

where $f = 2\gamma b \pm 2c_a(1 - \cos Q)$, $g = 4\gamma b \pm 4c_a - 2c_b \cos Q - \beta$, and the upper (lower) sign applies to $q = 0$ (π). The MI gain $G = |\mathrm{Im}\, K|$ is then given as

$$G = \frac{1}{\sqrt{2}} \left| \mathrm{Im}[-\alpha_2 \pm (\alpha_2^2 - 4\alpha_0)^{1/2}]^{1/2} \right|.$$

In the long-wavelength limit (small Q) the MI gain approaches that of the continuum model [27]. The maximum MI gain is plotted in Fig. 4 as a function of the stationary SH amplitude b and the mismatch β for in-phase $(q = 0)$ and out-of-phase $(q = \pi)$ solutions. With regard to the dispersion relation (3), these two cases correspond to opposite signs of dispersion of the linear waves. Evidently, the change of the character of dispersion critically affects the stability behavior. Although the stability ranges considerably differ for the two regimes, a common stable region with $(b < 0$ and $\beta < 0)$ can be identified. The consequences of the background stability properties for the dynamics of dark SLMs can be seen in Fig. 5. The propagation of two dark kinklike SLMs that are in close proximity in the b-β plane is displayed. The obviously stable SLM corresponds to the domain where $G = 0$ holds [Fig. 5(a)]. A slight change of the SH amplitude b causes the solution to move to the unstable region. As a

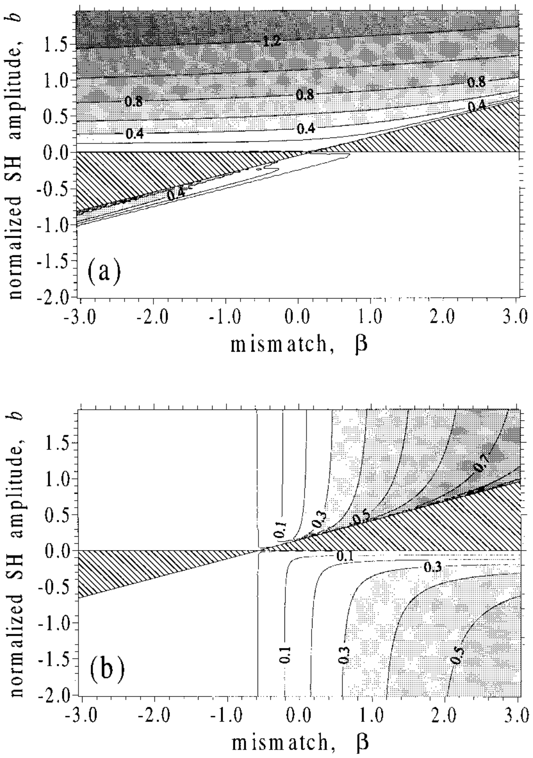

FIG. 4. Maximum MI gain of the background as a function of the normalized SH amplitude and the wave-vector mismatch. No plane-wave solution exists in the shaded region. Bright regions correspond to stable solutions ($G=0$). The parameters are $c_a=c_b=0.1$ and (a) $q=0$ and (b) $q=\pi$.

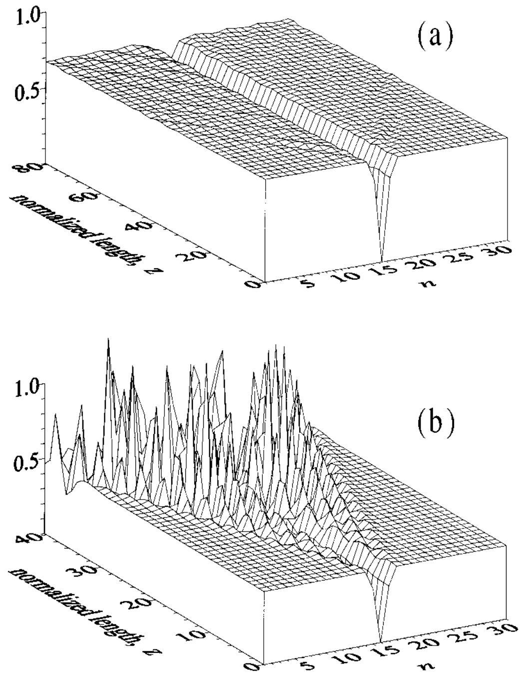

FIG. 5. Propagation of a dark, kinklike in-phase SLM (the normalized FF intensity is shown). The parameters are $c_a=c_b=0.1$, $\beta=-2$, and (a) $b=-0.76\rightarrow G_{\max}=0$ and (b) $b=-0.72\rightarrow G_{\max}=0.24$.

result, the dark SLM becomes unstable and decays after some distance [Fig. 5(b)] as do all SLMs that were termed ''exotic'' above.

In conclusion, families of bright as well as dark strongly localized modes in discrete quadratic media have been shown to exist. Among them are solutions that have no counterparts in discrete systems studied previously. A particular feature of SLMs in quadratic media consists in that FF solutions of different topologies form bound states with the same second-harmonic SLM. As far as bright SLMs are concerned, it has been shown that odd solutions are stable. This numerical finding agrees with the fact that the Peierls-Nabarro potential for this kind of solution attains a minimum. The stability of dark SLMs critically depends on the dispersion behavior of linear waves and the mismatch.

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